LETTERS

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A linear process in wall-bounded turbulent shear flows

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A linear process in wall-bounded turbulent shear flows has been investigated through numerical experiments. It is shown that the linear coupling term, which enhances non-normality of the linearized Navier–Stokes system, plays an important role in fully turbulent—and hence, nonlinear—flows. Near-wall turbulence is shown to decay without the linear coupling term. It is also shown that near-wall turbulence structures are not formed in their proper scales without the nonlinear terms in the Navier–Stokes equations, thus indicating that the formation of the commonly observed near-wall turbulence structures are essentially nonlinear, but the maintenance relies on the linear process. Other implications of the linear process are also discussed. © *2000 American Institute of Physics.* [S1070-6631(00)00708-X]

The transient growth due to non-normality of the eigenmodes of the linearized Navier–Stokes (N–S) equations has received much attention during the past several years (see, for example, Refs. 1–3). It has been shown that the energy of certain disturbances can grow to $\mathcal{O}(\mathrm{Re}^2)$ in time proportional to $\mathcal{O}(\mathrm{Re})$, where Re denotes Reynolds number of the flow.[2] It has been postulated that this transient growth, which is a linear process, can lead to transition to turbulence at a Reynolds number smaller than the critical Reynolds number, below which a classical linear stability theory based on the modal analysis predicts that all small disturbances decay asymptotically. As such, some investigators attributed this linear process as a possible cause for subcritical transition in some wall-bounded shear flows, such as plane Poiseuille flow and Couette flow.

Some investigators further postulated that the same linear process is also responsible for the observed wall-layer streaky structures in turbulent boundary layers.[1,4] The optimal disturbance, which has the largest transient growth according to their optimal perturbation theory, looks similar to the near-wall streamwise vortices that create the streaky structures in turbulent boundary layers. However, this optimal disturbance occupies the entire boundary layer, in contrast to the streamwise vortices in turbulent boundary layers, which are confined to the near-wall region. In order to relate their optimal perturbation theory to those structures observed in turbulent boundary layers, a time scale corresponding to the bursting process in turbulent boundary layers, which is essentially a nonlinear process, was introduced as an additional parameter.[4] It has been argued that the transient growth in turbulent boundary layers would be disrupted by turbulent motions on a time scale corresponding to the bursting process, which is smaller than the viscous time scale, and hence, the globally optimal disturbance will never have a chance to grow to its maximum possible amplitude. The notion that commonly observed wall-layer structures are related to a linear process, although it is the nonlinear process that determines the proper length scale, suggests that the same linear process may play an important role in fully nonlinear turbulent boundary layers.

Other evidence that a linear process may play an important role in turbulent boundary layers can be found in the work of Joshi *et al.*[5,6] and others,[7,8] who successfully applied controllers developed based on a liner system theory to the nonlinear flow in their attempt to reduce the viscous drag in turbulent boundary layers. Bewley[9] applied linear optimal control theory to a nonlinear convection problem. Although it is not clear how controllers based on a linearized model work so well for nonlinear flows and it is a subject of further investigation, these results suggest that the essential dynamics of near-wall turbulence may well be approximated by a linear model.

Motivated by the above findings, we investigate the role of this linear process in fully nonlinear turbulent flows. In particular, we investigate the role of the linear coupling term (see below for its definition), which is a source of the non-normality of the eigenmodes of the linearized Navier–Stokes equations, in wall-bounded shear flows, using a turbulent channel flow as an example.

In this Letter, we shall use (x,y,z) for the streamwise, wall-normal, and spanwise coordinates, respectively, and (u,v,w) for the corresponding velocity components. Reynolds number, Re_τ, is based on the wall-shear velocity, $u_\tau = \sqrt{\tau_w/\rho}$, and the channel half-width, h, where $\tau_w = \nu dU/dy|_w$ is the mean shear stress at the wall, and U, ν, and ρ denote the mean velocity, viscosity, and density, respectively. The superscript ''+'' denotes quantities nondimensionalized by ν and u_τ.

Representing the wall-normal velocity, v, and the wall-normal vorticity, ω_y, in terms of Fourier modes in the streamwise (x) and the spanwise (z) directions, the linearized N–S equations can be written in an operator form

$$\frac{d}{dt}\begin{bmatrix}\hat{v}\\ \hat{\omega}_y\end{bmatrix}=[A]\begin{bmatrix}\hat{v}\\ \hat{\omega}_y\end{bmatrix}, \tag{1}$$

where

$$[A]=\begin{bmatrix}L_{\mathrm{os}} & 0\\ L_{\mathrm{c}} & L_{\mathrm{sq}}\end{bmatrix}, \tag{2}$$

and the hat denotes a Fourier-transformed quantity. Here L_{os}, L_{sq}, and L_{c} represent the Orr–Sommerfeld, Squire, and the coupling operators, respectively, and defined as

$$\begin{aligned}L_{\mathrm{os}}&=\Delta^{-1}(-ik_xU\Delta+ik_x(d^2U/dy^2)+(1/\mathrm{Re})\,\Delta^2),\\ L_{\mathrm{sq}}&=-ik_xU+(1/\mathrm{Re})\,\Delta,\\ L_{\mathrm{c}}&=-ik_z(dU/dy),\end{aligned} \tag{3}$$

where k_x and k_z are the streamwise and spanwise wave numbers, respectively, $k^2=k_x^2+k_z^2$, $\Delta=\partial^2/\partial y^2-k^2$, and U is the mean velocity about which the N–S equations are linearized. Note that the full nonlinear N–S equations can be written also as

$$\frac{d}{dt}\begin{bmatrix}\hat{v}\\ \hat{\omega}_y\end{bmatrix}=[A]\begin{bmatrix}\hat{v}\\ \hat{\omega}_y\end{bmatrix}+\begin{bmatrix}\mathcal{N}_v\\ \mathcal{N}_{\omega_y}\end{bmatrix}, \tag{4}$$

where all nonlinear terms are lumped into $\mathcal{N}_v$ and $\mathcal{N}_{\omega_y}$. The operator A in this case, however, is a function of v and ω_y, because U depends on v and ω_y.

It has been shown that operator A in Eq. (2) is nonnormal, and hence, its eigenmodes are nonorthogonal, thus allowing a transient growth of energy even if all individual modes are stable and decay asymptotically.[1,3] Note that the coupling term L_{c} vanishes for two-dimensional (2-D) disturbances ($k_z=0$), and therefore, there is no coupling between v and ω_y for 2-D disturbances. For 3-D disturbances, however, v evolves independently, but ω_y is forced by v through the coupling term. It should be noted that L_{os} itself is not self-adjoint, and hence, 2-D disturbances can have a transient growth, but it was shown that 3-D disturbances have much larger transient growth due to the coupling term, which causes larger non-normality. In the present study, we concentrate on the role of the coupling term in fully nonlinear turbulent flows, using a fully developed turbulent channel flow as an example.

In order to investigate the role of the coupling term in fully turbulent flows, we proceed to solve the following modified nonlinear equations:

$$\frac{d}{dt}\begin{bmatrix}\hat{v}\\ \hat{\omega}_y\end{bmatrix}=\begin{bmatrix}L_{\mathrm{os}} & 0\\ 0 & L_{\mathrm{sq}}\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\omega}_y\end{bmatrix}+\begin{bmatrix}\mathcal{N}_v\\ \mathcal{N}_{\omega_y}\end{bmatrix}. \tag{5}$$

This modified system can be viewed as representing a synthetic turbulent flow without the coupling term, or a turbulent flow with control by which the coupling term is suppressed. For instance, surface blowing and suction activated to eliminate (reduce) the spanwise variation of v (i.e., $\partial v/\partial z$) could eliminate (reduce) the effect of the coupling term.

A spectral channel code similar to that of Kim *et al.*[10] was used to solve the above modified nonlinear equations. To further contrast the role of the coupling term, we used the modified N–S equations only in the upper half of the channel and the regular N–S equations in the lower half of the channel. We used the same Reynolds number ($\mathrm{Re}_\tau=100$) and grid ($32\times65\times32$ in x,y,z) as Lee *et al.*[11]

In the first numerical experiment, we used a regular turbulent velocity field obtained by Lee *et al*. as our initial field. Starting from this initial field, we integrated in time to see how the turbulent flow in the upper half of the channel evolves in the absence of the coupling term. Time evolution of the mean shear at both walls is shown in Fig. 1, which illustrates a drastic reduction in the wall shear without the coupling term. Several snapshots of the velocity field are shown in Fig. 2, where contours of streamwise vorticity in a

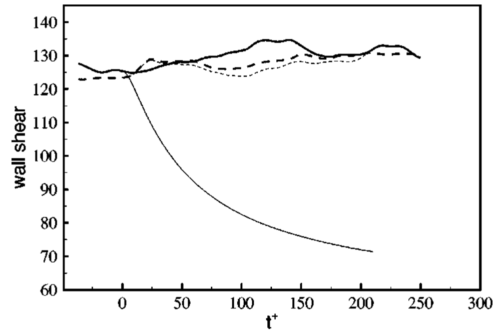

FIG. 1. Time evolution of mean shear at wall: ——, upper wall; — — —, lower wall. Thick lines are for a regular channel flow, while thin lines are for a channel flow with $L_{\mathrm{c}}=0$ in the upper half of the channel starting from $t^+=0$.

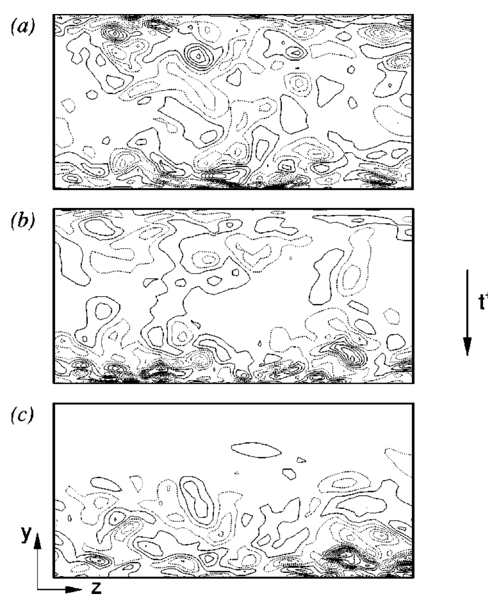

FIG. 2. Contours of streamwise vorticity in $y-z$ plane: (a) $t^+=0$; (b) $t^+=20$; (c) $t^+=200$. $-80<\omega_x<80$ with 18 contour levels. Note that $L_{\mathrm{c}}=0$ only in the upper-half of the channel.

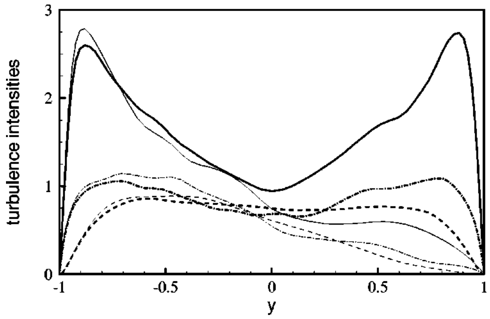

FIG. 3. Root-mean-square turbulence intensities: ——, $\sqrt{\overline{u^2}}$; - - - -, $\sqrt{\overline{v^2}}$; —·—·—, $\sqrt{\overline{w^2}}$. Thick lines are for $t^+=0$, while thin lines are for $t^+=180$.

$y-z$ plane are shown to illustrate the effect of the coupling term on turbulence structures. It is evident that streamwise vortices quickly disappear without the coupling term. The reduction of the wall shear in conjunction with the disappearance of the streamwise vortices is a common feature of many drag-reduced turbulent flows.[11] Turbulence intensities shown in Fig. 3 indicate drastic reductions without the coupling term.

In the second numerical experiment, we used an initial velocity field consisting of the same mean velocity as the first experiment but with random disturbances, and hence, there are no organized turbulence structures present initially. A divergence-free white-noise spectrum was used for this purpose. The amplitude was chosen such that neither they decay too quickly (too small) nor they cause a numerical instability due to non-smoothness of the initial condition (too

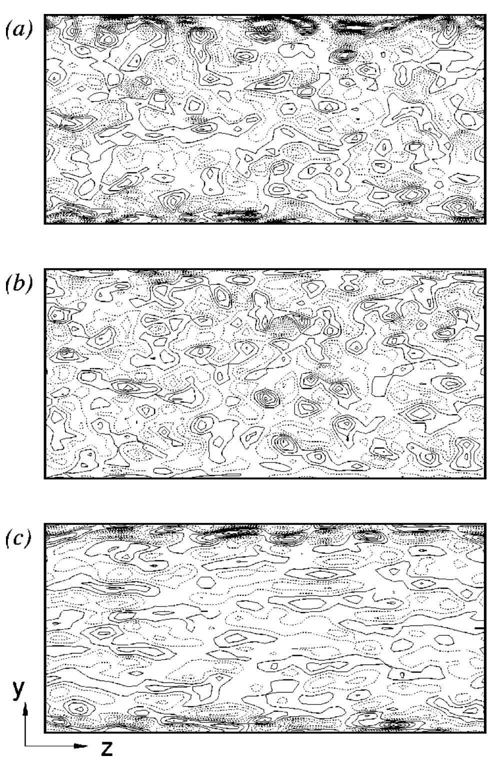

FIG. 4. Contours of streamwise vorticity in $y-z$ plane at $t^+=20$, started from an initial random field: (a) Case 1, regular turbulent flow; (b) Case 2, without the linear coupling term, L_c; (c) Case 3, without the nonlinear terms. Contour levels are the same as Fig. 2.

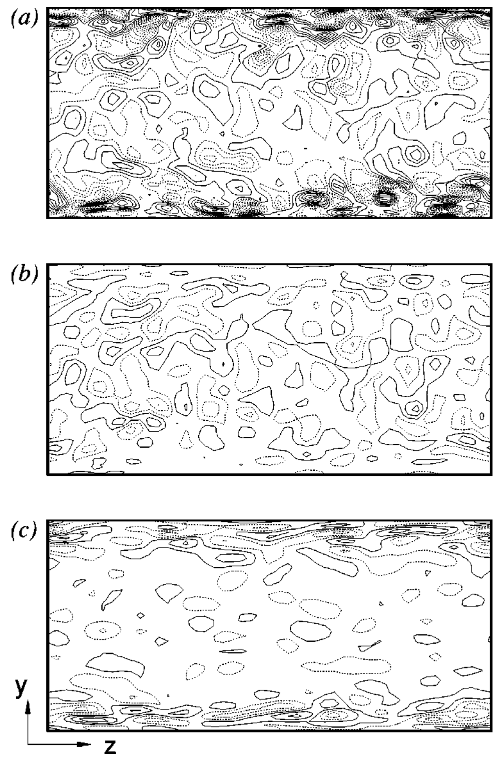

FIG. 5. Contours of streamwise vorticity in $y-z$ plane at $t^+=40$, started from a random initial field. See figure caption in Fig. 4 for legend.

large). Starting with the same random initial field, three different simulations were carried out: Case 1, with the full nonlinear N–S equations (i.e., regular turbulent flow); Case 2, with N–S equations without the linear coupling term; Case 3, with N–S equations without the nonlinear terms (i.e., linearized N–S). The purpose of these simulations is to investigate whether the linear coupling term is indeed responsible for formation of the streamwise vortices and near-wall streaks, and if so, whether the time scale associated with the formation of these structures corresponds to the bursting process ($t^+\approx 100$), as hypothesized by Butler and Farrell.[4]

Time evolution of the three velocity fields is shown in Figs. 4–6 with streamwise vorticity contours in a $y-z$ plane. Organized structures are discernible in all three cases as early as $t^+=20$ (Fig. 4), but they look different from each other. For Case 3 (without the nonlinear terms), the structures that appear from the structureless random initial condition have larger spanwise scales than those in the regular flow (Figs. 4 and 5). For Case 2 (without the linear coupling term), the vortical structures appear briefly (Fig. 4) but disappear quickly, especially in the wall region (Figs. 5 and 6), since they cannot be maintained without the linear coupling term as demonstrated in the first experiment mentioned above. For Case 1, the time for these structures to appear is shorter than that implied by the optimal disturbance mechanism of Butler and Farrell.[4] Note that the structures in Case 3 are already substantially different from those in Case 1 at $t^+=40$, indicating that the effect of nonlinear terms is felt much earlier than the eddy turnover time ($t^+\approx 100$) as proposed by Butler and Farrell.[4] The present result is also consistent with Jiménez and Pinelli,[12] who showed that the formation of streaky structures can be prevented by damping

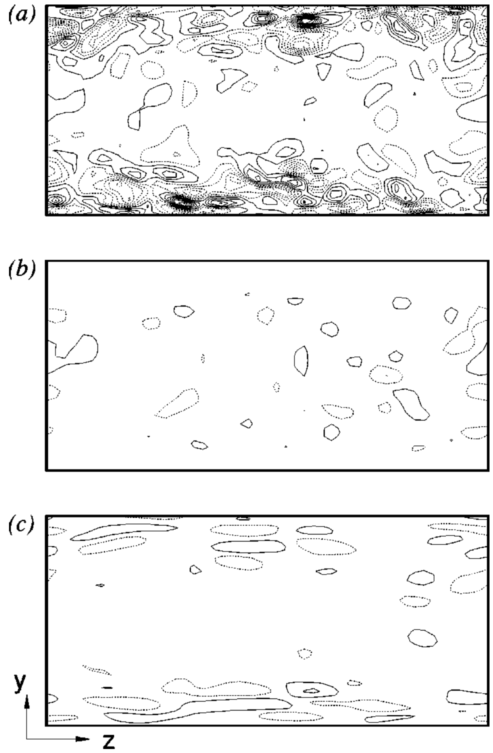

FIG. 6. Streamwise contours in $y-z$ plane at $t^+=80$, started from a random initial field. See figure caption in Fig. 4 for legend.

$\overline{(v\omega_z)}$ [the $\overline{(\ )}$ indicates streamwise average], which is related to the linear coupling term. While both results clearly demonstrate the essential role of the linear coupling term in the formation and maintenance of the wall-layer streaks, while the present work also indicates that a nonlinear mechanism is responsible for producing the proper streak spacing.

We have used several different initial conditions to determine whether the above results depend on initial conditions, but found no such evidence. It thus appears that both the nonlinear terms and the linear coupling term are necessary for the formation and maintaining of these structures at their proper scale. The nonlinear terms are necessary for the formation of streamwise vortices and the linear coupling term is necessary to generate the wall-layer streaks, the instability of which in turn strengthen the streamwise vortices through a nonlinear process. In the absence of either mechanism, turbulence ceases to exist. The result of this second experiment is consistent with Hamilton *et al.*[13] and Waleffe and Kim[14] in that the formation of the streamwise vortices is a result of a nonlinear process.

We have demonstrated that the linear process associated with the coupling term plays an important role even in fully nonlinear wall-bounded turbulent shear flows. Near-wall streamwise vortices, which play the essential role in the dynamics of wall-bounded shear flows, are seen to be formed but cannot be sustained without the coupling term.

This result is consistent with the analysis by Henningson and Reddy[15] who showed that non-normality of the linearized Navier–Stokes operator is a necessary condition for disturbances to grow for Reynolds number below the critical Reynolds number predicted by the traditional linear stability analysis. However, we believe this is the first direct demonstration that turbulence (nonlinear disturbance) decays when the non-normality of the underlying linear operator in nonlinear flows is reduced. The time scale associated with the formation is found to be smaller than the bursting process used in the optimal perturbation theory. The fact that the coupling term plays an essential role in maintaining the streamwise vortices, which have been found to be responsible for high skin-friction drag in turbulent boundary layers, suggests that an effective control algorithm for drag reduction should be aimed at reducing the effect of the coupling term in the wall region. In fact, the opposition control used by Choi *et al.*[16] can be viewed as a control scheme trying to reduce the effect of the coupling term by suppressing the spanwise variation of v in the wall region. It should be interesting to design a control algorithm that directly accounts for the coupling term in a cost function to be minimized.

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